

The Deviation of the Vacuum Refractive Index Induced by a Static Gravitational Field

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We analyzed the influence of static gravitational field on the vacuum and proposed the concept of inhomogeneous vacuum. According to the observational result of the light deflection in solar gravitational field as well as the corresponding Fermat's principle in the general relativity, we derived an analytical expression of the refractive index of vacuum in a static gravitational field. We found that the deviation of the vacuum refractive index is composed of two parts: one is caused by the time dilation effect, the other is caused by the length contraction effect. As an application, we simulated the effect of the gravitational lensing through computer programming and found that the missing central imaging could be interpreted in a reasonable way.

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I. INTRODUCTION

Vacuum is usually considered as “homogeneous” and “isotropic”, i.e., vacuum does not differ from place to place, and the refractive index of vacuum is always equal to 1. However, the recent theoretical and experimental progresses demonstrate that such concept of vacuum turns out to be inappropriate when there are matters or fields within finite distance. For example, the vacuum inside a microcavity is modified due to the existence of the cavity mirrors, which will alter the zero-point energy inside the cavity and cause an attractive force between the two mirrors known as Casimir effect [1, 2], which has been verified experimentally [3, 4]. A second example is that, under the influence of electromagnetic field, vacuum can be polarized, which has led to astonishingly precise agreement between predicted and observed values of the electron magnetic moment and Lamb shift, and may influence the motion of photons [5]. Dupays *et al.* [6] studied the propagation of light in the neighborhood of magnetized neutron stars. They pointed out that the light emitted by background astronomical objects will be deviated due to the optical properties of quantum vacuum in the presence of a magnetic field. Also in [7], Rikken and Rizzo considered the anisotropy of the optical properties of the vacuum when a static magnetic field \mathbf{B}_0 and a static electric field \mathbf{E}_0 are simultaneously applied perpendicular to the direction of light propagation. They predicted that magnetoelectric birefringence will occur in vacuum under such conditions. They also demonstrated that the propagation of light in vacuum becomes anisotropic with the anisotropy in the refractive index being proportional to $\mathbf{B}_0 \times \mathbf{E}_0$.

The facts that the propagation of light in vacuum can be modified by applying electromagnetic fields to the vacuum implies that the vacuum is actually a special kind of optical medium [5, 6]. This is similar to the Kerr electro-optic effect and the Faraday magneto-optic effect in nonlinear dielectric medium. This similarity between the vacuum and the dielectric medium implies that vacuum must also have its inner structure, which could be influenced by matter or fields as

well. Actually, the structure of quantum vacuum has already been investigated in a number of papers [8, 9, 10].

In this paper, with the analysis of the influence of static gravitational field on the vacuum, we put forward a new concept that the curved spacetime around a certain matter can be treated as an optical medium with a graded refractive index. We suggest that the so-called curved spacetime is a reflection of the vacuum inhomogeneity caused by the influence of gravitational matter. Based on this idea, the refractive index of vacuum is derived. We will also apply this concept to un-puzzle the problem of the central image missing in almost all the observed cases of gravitational lensing [11].

II. THE DEVIATION OF THE VACUUM REFRACTIVE INDEX

According to the astronomical observation, the light propagating through a space with a celestial body nearby will be deflected. It can be interpreted with the curved spacetime in general relativity. As a matter of fact, it can also be interpreted with the assumption that the vacuum around matter is inhomogeneous with refractive index deviated from 1. Here we put forward a theoretical model to describe the refractive index profile based on the Fermat's principle for the propagation of light in a static gravitational field, which was given by Landau and Lifshitz [12]:

$$\delta \int g_{00}^{-1/2} dl = 0, \quad (1)$$

where dl is the local length element passed by light and measured by the observer at position r in the gravitational field, r is the distance from this element of light to the center of gravitational matter M , g_{00} is a component of the metric tensor $g_{\mu\nu}$, $g_{00}^{-1/2} dl$ corresponds to an element of optical path length. $g_{00}^{-1/2} = dt/d\tau$, where $d\tau$ represents the time interval measured by the local observer for a light ray passing through the length dl , while dt is the corresponding time measured by the

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observer at infinity. Eq.(1) could then be rewritten as

$$\begin{aligned}\delta \int g_{00}^{-1/2} dl &= \delta \int \frac{dt}{d\tau} dl \\ &= \delta \int \frac{dt}{d\tau} \frac{dl}{ds} ds \\ &= \delta \int n ds = 0,\end{aligned}\quad (2)$$

where ds is the length element measured by the observer at infinity, corresponding to the local length dl .

Eq.(2) shows that if we set the scale of length and time at infinity as a standard scale for the whole gravitational space and time, the propagation of light then satisfies the standard representation of Fermat's principle, with the space — actually the vacuum — possessing a refractive index given by

$$n = \frac{dt}{d\tau} \frac{dl}{ds} = n_1 n_2. \quad (3)$$

The factor n_1 of the refractive index relating to the time transformation effect $dt/d\tau$ can be derived from the Newtonian attraction, which contributes partially to the deflection of light. Considering a photon of relativistic mass m_∞ at the infinity moving down to position r , the work done on to the photon by the Newtonian gravity is

$$-\frac{GMm}{r^2} dr = d(mc^2), \quad (4)$$

where G is the gravitational constant, c is the velocity of light, M is the mass of a star (say the Sun), r is the distance to the center of the star. Integrating Eq.(4) gives

$$m_r = m_\infty e^{\frac{GM}{rc^2}}, \quad (5)$$

where m_r is the relativistic mass of the photon at position r .

Since the photon energy is $E = hv = mc^2$, where h is the Planck constant, v is the photon frequency, then we have $m = hv/c^2$. Substituting it into Eq.(5) gives

$$v_r = v_\infty e^{\frac{GM}{rc^2}}. \quad (6)$$

It is just the frequency shift caused by the gravitational force, which reflects that a clock in a gravitational field runs slower than that far away from the gravitational center. That is

$$d\tau = e^{-\frac{GM}{rc^2}} dt, \quad (7)$$

where $d\tau$ denotes the time measured by a clock at position r , dt is the converted time of $d\tau$, i.e., the time measured by the clock at infinity. This relation indicates that, if the length scale is the same, i.e., $dl = ds$, when an observer at position r reports a light velocity $c_1 = dl/d\tau$, it should be converted by the observer at infinity into

$$c'_1 = \frac{ds}{dt} = \frac{dl}{e^{\frac{GM}{rc^2}} dt} = c_1 e^{-\frac{GM}{rc^2}}. \quad (8)$$

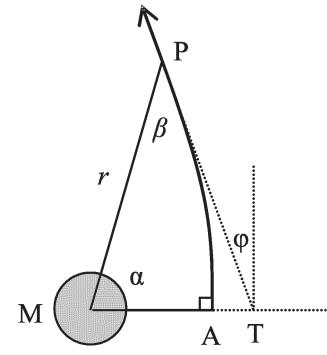


FIG. 1: Light deflection caused by a graded refractive index.

This change of light velocity will certainly bring a deflection to the light propagation. The corresponding refractive index is

$$n_1 = \frac{c_1}{c'_1} = \frac{dt}{d\tau} = e^{\frac{GM}{rc^2}}. \quad (9)$$

Let us now consider the deflection angle caused by this graded refractive index. In Fig.1, the curve AP represents the light ray, β is the angle between the position vector \mathbf{r} and the tangent at the point P on the ray, φ is the deflection angle of light. Since the refractive index shown in Eq.(9) has spherical symmetry, i.e., depends only on the distance r for a given mass M , according to the Fermat's principle

$$\delta \int n ds = 0, \quad (10)$$

where $ds = dr \sqrt{1 + (r\dot{\alpha})^2}$, $\dot{\alpha} = d\alpha/dr$, $n = n(r)$, we have the corresponding Lagrangian function

$$L(\alpha, \dot{\alpha}; r) = n(r) \sqrt{1 + (r\dot{\alpha})^2}. \quad (11)$$

Using the Lagrangian equation

$$\frac{d}{dr} \left(\frac{\partial L}{\partial \dot{\alpha}} \right) - \frac{\partial L}{\partial \alpha} = 0, \quad (12)$$

we get [13]

$$nr \sin \beta = \text{constant}, \quad (13)$$

or

$$nr \sin \beta = n_0 r_0, \quad (14)$$

where r_0 and n_0 represent the radius and refractive index at the nearest point A respectively.

Since

$$\tan \beta = \frac{r d\alpha}{dr}, \quad (15)$$

associating with Eq.(14) reaches

$$d\alpha = \frac{dr}{r \sqrt{(\frac{nr}{n_0 r_0})^2 - 1}}. \quad (16)$$

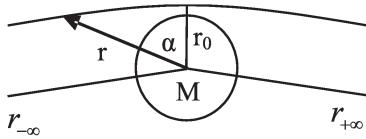


FIG. 2: Light deflection in solar gravitational field.

For a light ray passing by the Sun as shown in Fig.2, the total angular displacement of the radius vector \mathbf{r} reads

$$\Delta\alpha = 2 \int_{r_0}^{\infty} \frac{dr}{r \sqrt{(\frac{nr}{n_0 r_0})^2 - 1}}, \quad (17)$$

where r_0 represents the nearest distance to the center of the Sun. Because the gravitational field of the Sun is a weak field, the value of $GM/r c^2$ is quite small, so substituting Eq.(9) into Eq.(17) gives a solution of first order approximation

$$\Delta\alpha = \pi + \frac{2GM}{r_0 c^2}. \quad (18)$$

Then the total deflection angle of light caused by the refractive index n_1 in solar gravitational field is

$$\Delta\varphi_1 = \Delta\alpha - \pi = \frac{2GM}{r_0 c^2}. \quad (19)$$

In fact, this result was obtained early in 1911 by Einstein [14], who also investigated the effect of red shift and the corresponding slowing down of the light velocity in gravitational field and then figured out the light deflection as shown in Eq.(19) with the use of Huygens' principle. Since the actual total deflection angle of light propagation calculated by the general relativity [15, 16] and measured by the astronomical observation [17] is twice that value, we then come to know that the length transformation effect dl/ds in Eq.(2) must have the same relation as that of the time transformation effect $dt/d\tau$ expressed in Eq.(9), namely

$$\frac{dl}{ds} = e^{-\frac{GM}{rc^2}}. \quad (20)$$

This relation indicates that a ruler in a gravitational field is shorter than that far away from the gravitational center. So when an observer at position r reports a length dl , it should be converted by the observer at infinity into

$$ds = e^{-\frac{GM}{rc^2}} dl. \quad (21)$$

For a light passing through a length dl , if the time scale is the same, i.e., $d\tau = dt$, the light velocity $c_2 = dl/d\tau$ reported by the observer at position r should then be converted by the observer at infinity into

$$c'_2 = \frac{ds}{dt} = \frac{e^{-\frac{GM}{rc^2}} dl}{d\tau} = c_2 e^{-\frac{GM}{rc^2}}. \quad (22)$$

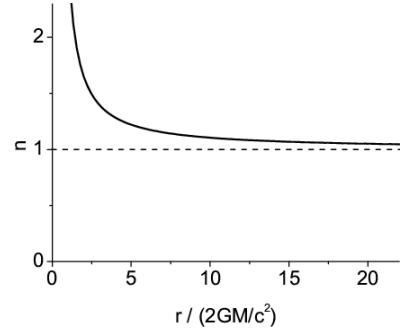


FIG. 3: The dependence of the vacuum refractive index n on the distance r .

This change of light velocity will also bring a deflection to the light propagation. The refractive index corresponding to this kind of deflection is

$$n_2 = \frac{c_2}{c'_2} = \frac{dl}{ds} = e^{\frac{GM}{rc^2}}, \quad (23)$$

which also causes a deflection angle of light

$$\Delta\varphi_2 = \frac{2GM}{r_0 c^2}. \quad (24)$$

Therefore, the total deflection angle of light in solar gravitational field is

$$\Delta\varphi = \Delta\varphi_1 + \Delta\varphi_2 = \frac{4GM}{r_0 c^2}. \quad (25)$$

The above result shows that, if the two refraction effects are considered simultaneously, then the gravitational space — actually the vacuum in the gravitational field — can be regarded as an optical medium with a total refractive index given by

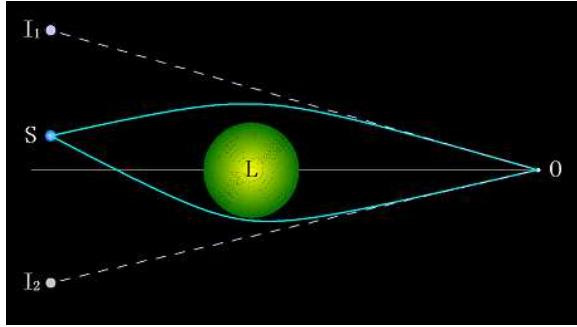
$$n = n_1 n_2 = e^{\frac{2GM}{rc^2}}. \quad (26)$$

n is composed of two factors: n_1 — related with the time transformation or “curved time”; n_2 — related with the space transformation or “curved space”. So the curved spacetime of general relativity is reflected in the synthesized refractive index n , which is also a reflection of the inhomogeneity of the vacuum, showing that the vacuum near the matter is influenced more than that far away from the matter.

The above expression of n shows that the refractive index of the vacuum at the infinity from the gravitational matter is 1, i.e., the usual refractive index of vacuum. The closer of the position to the center of matter M , the higher the refractive index of the vacuum. The relation between n and r is depicted in Fig.3, where $2GM/c^2$ is taken as the unit of r . For examples, the corresponding radii for the surface of the Sun in solar gravitational field and the surface of the Earth in earth gravitational field are 2.36×10^5 and 7.20×10^8 respectively — both are far beyond the r -axis illustrated in Fig.3.

The deviation of the vacuum refractive index from the usual value 1 is given by

$$\Delta n = n - 1 = e^{\frac{2GM}{rc^2}} - 1. \quad (27)$$

FIG. 4: The two images I_1, I_2 of a gravitational lens.

In weak field it becomes

$$\Delta n = \frac{2GM}{rc^2}. \quad (28)$$

In order to provide the readers with a quantity impression, let us give two examples. For the solar gravitational field ($M = 1.99 \times 10^{30} kg$), the deviation of n on the surface of the Sun ($r = 6.96 \times 10^8 m$) is 4.24×10^{-6} . For the earth gravitational field ($M = 5.98 \times 10^{24} kg$), the deviation of n on the surface of the Earth ($r = 6.38 \times 10^6 m$) is only 1.39×10^{-9} , which is so small that it can hardly be observed in usual experiments. Nevertheless, for a massive celestial body such as a heavy star, a galaxy or a cluster of galaxies, the deviation is not only observable, but also important and useful in gravitational astronomy.

III. APPLICATIONS

The deflection of light by massive bodies leads to the effect of gravitational lensing. Formerly, this effect should be calculated complicatedly with the general relativity [18]. Once we have introduced the concept of graded vacuum refractive index and obtained its relation with mass M and position r , the problem of gravitational lensing could then be treated easily with the conventional optical method.

Considering a source S and a lens L of mass M , the light emitted from S is bent due to the gravitational field of the lens. The bent light could be figured out through Eq.(13) and Eq.(26). Drawing the extension line of the light from the observer O , the apparent (observed) position of the source image I could then be found out. The result is shown in Fig.4.

This method could also be applied in studying the central imaging. In doing this, the vacuum refractive index profile inside the lensing body should be considered as well.

Noticing that Eq.(26) could be virtually rewritten as

$$n = e^{-\frac{2P_r}{c^2}}, \quad (29)$$

where P_r represents the gravitational potential at position r from the center of the lens.

As a model for discussion, we suppose a lens (for example, a galaxy or a cluster of galaxies) of radius R with a density

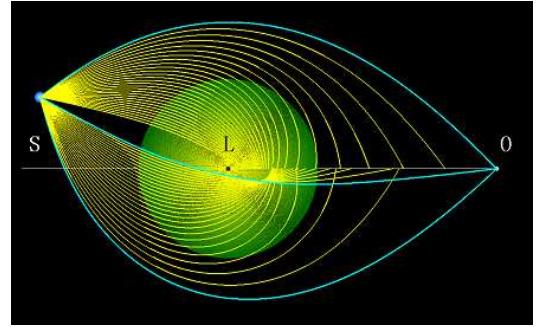


FIG. 5: A ray tracing result for the central imaging.

distribution

$$\rho = \rho_0 [1 - (\frac{r}{R})^k], \quad (30)$$

where ρ_0 is the central density of the lens, $0 \leq r \leq R$, $k > 0$. The density ρ decreases with the distance r from the center of mass; the decreasing varies with the parameter k . This model gives the distribution of gravitational potential as

$$P_o = -4\pi\rho_0 G \frac{k}{3(3+k)} \frac{R^3}{r},$$

$$P_i = -4\pi\rho_0 G \{ \frac{k}{2(2+k)} R^2 - [\frac{1}{6} - \frac{1}{(2+k)(3+k)} (\frac{r}{R})^k] r^2 \}, \quad (31)$$

for outside ($r \geq R_0$) and inside ($r \leq R_0$) the gravitational lens respectively.

The vacuum refractive index profile outside and inside the gravitational lens then reads

$$n_o = \exp \left[\frac{8\pi\rho_0 G}{c^2} \frac{k}{3(3+k)} \frac{R^3}{r} \right];$$

$$n_i = \exp \left\{ \frac{8\pi\rho_0 G}{c^2} \{ \frac{k}{2(2+k)} R^2 - [\frac{1}{6} - \frac{1}{(2+k)(3+k)} (\frac{r}{R})^k] r^2 \} \right\} \quad (32)$$

Fig.5 shows a ray tracing result for the imaging of a gravitational lens with the above described vacuum refractive index profile. In the figure, only three paths of ray (the three thick lines) could pass through the observer O , forming the upper, lower and central images respectively. From the figure, we find that, under the same conditions, the larger the distance OL from the observer to the lens, the closer the central imaging light to the center of the lens. If the source S and the observer O are counterchanged, it could also be known from the figure that, the larger the distance SL from the source to the lens, the closer the central imaging light to the center of the lens. In addition, through the change of the lens mass $M = \frac{4}{3}\pi R^3 \rho_0 k / (3+k)$, we also find that, when the mass M increases, the distance from the central imaging light to the center of the lens decreases (Fig.6, where the mass ratio of the lenses corresponding to the four central imaging rays from bottom to top is $2 : 3 : 4 : 5$).

For the actual condition of gravitational imaging, the distances OL , SL and the mass M are all astronomical figures;

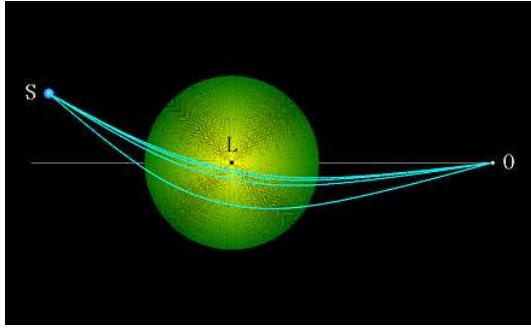


FIG. 6: Tracing the central imaging rays for lenses of different mass.

therefore, the light of central imaging is extremely close to the center of the lens. However, for a lensing body with a density increasing towards the center, it is possible that there are barrier matters near the center which will destroy the formation of the central image. Besides, the relatively longer inner path of the central imaging light adds the possibility of light being held back by the lens matters on the way. These and some other factors such as the relative faintness of the central imaging light and the possibly higher brightness of the lens core itself, all decrease the possibility of central imaging being actually observed. This analysis is firmly supported by the fact that the number of observed images is not “odd” as expected by the existed theories but “even” in almost all cases of gravitational lensing [11].

We have proposed the concept of inhomogeneous vacuum with graded refractive index based on the analysis of the influence of static gravitational field on the vacuum. we derived the expression of this refractive index analytically. By using this expression, we investigated the effect of gravitational lensing in a conventional optical way and provided a reasonable interpretation for the problem of central image missing.

The result indicates that, the concept of inhomogeneous vacuum is mathematically equivalent to the curved spacetime in the general relativity; therefore, an effective and convenient alternative method (i.e., optical method) could be established to solve the so complicated problems in gravitational astronomy. Physically, under such point of view, the motion of light in gravitational space is a motion of light wave in a quantum vacuum with graded refractive index. And as we know that, in conventional optics, the Fermat’s principle says that the optical path between two given points is an extremum. This is also equivalent to the theorem in the general relativity that a particle always moves along a geodesic line in a curved space-time.

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